

# On Bohr-Sommerfeld-Heisenberg Quantization\*

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## Abstract

This paper presents the theory of Bohr-Sommerfeld-Heisenberg quantization of a completely integrable Hamiltonian system in the context of geometric quantization. The theory is illustrated with several examples.

## 1 Introduction

Most texts on quantum mechanics have a short section on the old quantum theory. They discuss Bohr's quantization of the harmonic oscillator and Sommerfeld's results on the energy spectrum of the hydrogen atom. Usually they mention of Heisenberg's quantum mechanics and give a description of Schrödinger's wave mechanics. Schrödinger's theory is further discussed in the framework of the modern quantum mechanics. Heisenberg's theory is relegated to a criptic remark that Dirac proved that that the theories of Heisenberg and of Schrödinger are equivalent. In [8], Dirac showed that Heisenberg's matrices can be also obtained in the Schrödinger theory, but he did not state that these theories give the same physical results.

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Geometric quantization provides an explanation of Dirac's theory in the framework of modern differential geometry. Within geometric quantization, it is easy to understand Bohr-Sommerfeld quantization rules; I discussed them in my book [11]. However, for a long time I could not fit Heisenberg's matrix mechanics into the framework of geometric quantization.

The breakthrough came when Richard Cushman explained to me the notion of quantum monodromy introduced in a joint paper with Duistermaat [4]. If a Hamiltonian system with  $n$ -degrees of freedom admits a global Hamiltonian action of the  $n$ -torus, then the Bohr-Sommerfeld conditions are global and they define the structure of an  $n$ -dimensional lattice on the corresponding basis of the space of quantum states. Cushman and Duistermaat showed that, in the presence of classical monodromy, this lattice structure was only local.

In this lecture, I will describe our understanding of Heisenberg's quantum mechanics within the framework of geometric quantization. I do not know if our approach has any relation to Heisenberg's ideas. However, I hope to convince you that we obtain a well defined quantum theory consistent with the principles of geometric quantization. More precisely, the theory we obtain generalizes geometric quantization, as formulated by Kostant, to the case of a singular polarization.

## 2 Completely integrable systems

Let  $(P, \omega)$  be a symplectic manifold of dimension  $2n$ . We consider a completely integrable system on  $(P, \omega)$  with action angle coordinates  $(A_i, \varphi_i)$  defined on an open dense subset  $U$  of  $P$ . The symplectic form  $\omega$  restricted to  $U$  is  $\omega|_U = d\theta$ , where  $\theta = \sum_{i=1}^n d(A_i d\varphi_i)$ .

**Assumption 1** We assume that the action coordinates  $A_i$  are globally defined on  $P$ .

This implies that we have a symplectic action of the the torus group  $\mathbb{T}^n$  with the momentum map  $J : P \rightarrow \mathbb{R}^n : p \mapsto J(p) = (A_1(p), \dots, A_n(p))$ , where we have identified the Lie algebra of  $\mathbb{T}^n$  with  $\mathbb{R}^n$ .

### 3 Bohr-Sommerfeld Quantization

The Hamiltonian vector field  $X_f$  of a function  $f \in C^\infty(P)$  is defined by  $X_f \lrcorner \omega = -df$ , where  $\lrcorner$  is the left interior product (contraction on the left).

For each  $i = 1, \dots, n$ , the Hamiltonian vector field  $X_{A_i}$  generates the action on  $P$  of the  $i^{\text{th}}$  component  $\mathbb{T}_i$  of the torus group  $\mathbb{T}^n = \mathbb{T} \times \mathbb{T} \times \dots \times \mathbb{T}$ . We denote by  $O_{i,p}$  the orbit of  $\mathbb{T}_i$  through  $p \in P$ . Clearly,  $A_i$  is constant on each orbit  $O_{i,p}$ .

**Bohr-Sommerfeld Quantization Rule** For each  $i = 1, \dots, n$ , the quantum spectrum of  $A_i$  consists of the values  $A_i(p)$  on orbits  $O_{i,p}$  satisfying the condition

$$\int_{O_{i,p}} A_i d\varphi_i = m_i h, \quad (1)$$

where  $m_i$  is the integer and  $h$  denotes Planck's constant.

Integrating equation (1), we conclude that the quantum spectrum of the  $i^{\text{th}}$  action is given by

$$A_i = m_i \hbar, \quad (2)$$

where  $\hbar$  is Planck's constant divided by  $2\pi$ .

**Assumption 2** For each n-tuple  $\mathbf{m} = (m_1, \dots, m_n)$  of integers, the set

$$\mathbb{T}_{\mathbf{m}} = \{p \in P \mid A_i(p) = m_i \hbar \ \forall \ i = 1, \dots, n\}. \quad (3)$$

is connected.

Under this assumption,  $\mathbb{T}_{\mathbf{m}}$  is a torus. Otherwise, it would be the union of disjoint tori, and we would have to introduce an additional index to label connected components. In the following, we shall refer to sets  $\mathbb{T}_{\mathbf{m}}$  defined by equation (3) as Bohr-Sommerfeld tori.

### 4 Link to Geometric Quantization

Suppose that we want to perform geometric quantization of our completely integrable system in the real polarization  $D$  spanned by the Hamiltonian vector fields  $X_{A_i}$  of the momenta  $A_1, \dots, A_n$ .

Let  $L$  be a prequantization line bundle of  $(P, \omega)$ . Thus,  $L$  is a complex line bundle over  $P$ , with a connection  $\nabla$  such that

$$(\nabla_{X_1} \nabla_{X_2} - \nabla_{X_2} \nabla_{X_1} - \nabla_{[X_1, X_2]})\sigma = -\frac{i}{\hbar}\omega(X_1, X_2)\sigma$$

for each section  $\sigma$  of  $L$  and every pair  $X_1, X_2$  of vector fields on  $P$ .

The quantum states of the system are given by sections  $\sigma$  of  $L$  that are covariantly constant along the polarization  $D$ . If  $\Lambda$  is a leaf of  $D$ , it is a torus, and the restriction  $\sigma|_\Lambda$  of a section  $\sigma$  of  $L$  that is covariantly constant along  $D$  vanishes unless the holonomy group of the restriction of  $\nabla$  to  $\Lambda$  vanishes.

**Proposition** The holonomy group of the restriction of  $\nabla$  to a leaf  $\Lambda$  of  $D$  vanishes if and only if  $\Lambda$  satisfies the Bohr-Sommerfeld conditions; that is  $\Lambda = \mathbb{T}_{\mathbf{m}}$  for some  $n$ -tuple  $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$ .

The proof of this proposition can be found in reference [11].  $\square$

For a completely integrable system, tori  $\mathbb{T}_{\mathbf{m}}$  are submanifolds of  $P$  of codimension at least  $n$ , and the only smooth section of  $L$  that is covariantly constant along  $D$  is the zero section. However, we may identify quantum states with distribution sections of  $L$  that are smooth and covariantly constant along leaves of  $D$ . Under this interpretation of quantum states, to every non-empty Bohr-Sommerfeld torus  $\mathbb{T}_{\mathbf{m}}$  in  $P$ , we can associate a non-zero distribution section  $\sigma_{\mathbf{m}}$  of  $L$  with support in  $\mathbb{T}_{\mathbf{m}}$ , and such that the restriction  $\sigma_{\mathbf{m}}|_{\mathbb{T}_{\mathbf{m}}}$  of  $\sigma_{\mathbf{m}}$  to  $\mathbb{T}_{\mathbf{m}}$  is a smooth covariantly constant section the restriction of  $L$  to  $\mathbb{T}_{\mathbf{m}}$ . On the space  $\mathfrak{S}$  of distribution sections of  $L$  that are spanned by sections  $\sigma_{\mathbf{m}}$  introduce a scalar product  $\langle \cdot | \cdot \rangle$  such that

$$\langle \sigma_{\mathbf{m}} | \sigma_{\mathbf{m}'} \rangle = \delta_{\mathbf{m}\mathbf{m}'} = \delta_{m_1 m'_1} \dots \delta_{m_n m'_n}. \quad (4)$$

For each Bohr-Sommerfeld torus  $\mathbb{T}_{\mathbf{m}}$ , the section  $\sigma_{\mathbf{m}}$  introduced above is defined by  $\mathbb{T}_{\mathbf{m}}$  up to an arbitrary non-zero complex factor. Therefore, the collection  $\{\mathbb{T}_{\mathbf{m}}\}$  of all Bohr-Sommerfeld tori in  $P$  determines only the orthogonality property of basic vectors  $\sigma_{\mathbf{m}}$ . For a covariantly constant section  $\sigma$  with support in  $\mathbb{T}_{\mathbf{m}}$ , the norm  $\|\sigma\|$  depends on the choice of the basic section  $\sigma_{\mathbf{m}}$ .

We denote by  $\mathfrak{H}$  the Hilbert space obtained by the completion of  $\mathfrak{S}$  in the norm given by  $\langle \cdot | \cdot \rangle$ .  $\mathfrak{H}$  is our space of quantum states. To each function

$f \in C^\infty(P)$ , such that  $f = F(A_1, \dots, A_n)$  for some  $F \in C^\infty(\mathbb{R}^n)$ , the Bohr-Sommerfeld quantization associates the quantum operator  $\mathbf{Q}_f$  on  $\mathfrak{H}$  such that, for every basic section  $\sigma_{\mathbf{m}}$ ,

$$\mathbf{Q}_f \sigma_{\mathbf{m}} = F(\mathbf{m}\hbar) \sigma_{\mathbf{m}}.$$

It follows from Assumption 2 that the spectrum of the action operators  $\mathbf{Q}_{A_i}$  is simple.

A shortcoming of the Bohr-Sommerfeld quantization is that it is defined only on the commutative algebra consisting of smooth functions of the actions. In particular, Bohr-Sommerfeld quantization does not allow for quantization of any function of the angles. Moreover, it leads only to diagonal operators in  $\mathfrak{H}$ .

## 5 Shifting operators

Bohr-Sommerfeld conditions together with Assumptions 1 and 2 imply that the basis  $\{\sigma_{\mathbf{m}}\}$  is a lattice. Therefore, there are well defined operators corresponding to shifting along the generators of the lattice.

For each  $i = 1, \dots, n$ , let

$$\mathbf{m}_i = \{m_1, \dots, m_{i-1}, m_i - 1, m_{i+1}, \dots, m_n\}$$

and

$$\mathbf{m}^i = \{m_1, \dots, m_{i-1}, m_i + 1, m_{i+1}, \dots, m_n\}.$$

We define shifting operators  $\mathbf{a}_i$  on  $\mathfrak{H}$  by

$$\mathbf{a}_i \sigma_{\mathbf{m}} = \begin{cases} \sigma_{\mathbf{m}_i} & \text{if } \mathbb{T}_{\mathbf{m}_i} \neq \emptyset \\ 0 & \text{if } \mathbb{T}_{\mathbf{m}_i} = \emptyset \end{cases}. \quad (5)$$

The adjoint operators  $\mathbf{a}_i^\dagger$  are given by

$$\mathbf{a}_i^\dagger \sigma_{\mathbf{m}} = \begin{cases} \sigma_{\mathbf{m}^i} & \text{if } \mathbb{T}_{\mathbf{m}^i} \neq \emptyset \\ 0 & \text{if } \mathbb{T}_{\mathbf{m}^i} = \emptyset \end{cases}. \quad (6)$$

**Proposition 1** The shifting operators satisfy the following commutation relations

$$[\mathbf{a}_k, \mathbf{Q}_{A_j}] = \delta_{kj} \hbar \mathbf{a}_k, \quad (7a)$$

$$[\mathbf{a}_k^\dagger, \mathbf{Q}_{A_j}] = -\delta_{kj} \hbar \mathbf{a}_k^\dagger. \quad (7b)$$

The Poisson bracket relations between actions and angles are

$$\{e^{-i\varphi_k}, A_j\} = -i\delta_{kj}e^{-i\varphi_k}.$$

Hence, Dirac's quantization conditions

$$[\mathbf{Q}_{f_1}, \mathbf{Q}_{f_2}] = i\hbar \mathbf{Q}_{\{f_1, f_2\}} \quad (8)$$

suggest the identification  $\mathbf{a}_k = \mathbf{Q}_{e^{-i\varphi_k}}$  and  $\mathbf{a}_k^\dagger = \mathbf{Q}_{e^{i\varphi_k}}$ , where  $\varphi_k$  is the angle coordinate corresponding to the action  $A_k$ , provided the functions  $e^{-i\varphi_k}$  and  $e^{i\varphi_k}$  are globally defined on  $P$ .

## 6 Heisenberg Quantization

Since not all sets  $\mathbb{T}_{\mathbf{m}}$  are  $n$ -tori, we cannot expect that all exponential functions  $e^{-i\varphi_k}$  are globally defined. We can try to replace  $e^{-i\varphi_k}$  by a globally defined smooth function  $f_k$  of the form  $\chi_k = r_k e^{-i\varphi_k}$ , where the coefficient  $r_k$  depends only on the actions and vanishes at the points at which  $e^{i\varphi_k}$  is not defined. In the following we shall refer to functions  $\chi_k$  as Heisenberg functions.

We have the following Poisson bracket relations

$$\{\chi_k, A_j\} = -i\delta_{kj}\chi_k \text{ and } \{\bar{\chi}_k, A_j\} = i\delta_{kj}\bar{\chi}_k. \quad (9)$$

By Dirac's quantization conditions, we get

$$[\mathbf{Q}_{\chi_k}, \mathbf{Q}_{A_j}] = \delta_{kj}\hbar \mathbf{Q}_{\chi_k}, \quad (10)$$

$$[\mathbf{Q}_{\bar{\chi}_k}, \mathbf{Q}_{A_j}] = -\delta_{kj}\hbar \mathbf{Q}_{\bar{\chi}_k}. \quad (11)$$

For each basic vector  $\sigma_{\mathbf{m}}$  of  $\mathfrak{H}$ ,

$$\begin{aligned} \mathbf{Q}_{A_j}(\mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}}) &= \mathbf{Q}_{\chi_j}(\mathbf{Q}_{A_j}\sigma_{\mathbf{m}}) - [\mathbf{Q}_{\chi_j}, \mathbf{Q}_{A_j}]\sigma_{\mathbf{m}} \\ &= \mathbf{Q}_{\chi_j}(\hbar m_j \sigma_{\mathbf{m}}) - \hbar \mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}} \\ &= \hbar(m_j - 1)\mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}}. \end{aligned} \quad (12)$$

Thus,  $\mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}}$  is proportional to  $\sigma_{\mathbf{m}_j}$ . A similar argument shows that  $\mathbf{Q}_{\bar{\chi}_j}\sigma_{\mathbf{m}}$  is proportional to  $\sigma_{\mathbf{m}^j}$ . Hence,  $\mathbf{Q}_{\chi_j}$  and  $\mathbf{Q}_{\bar{\chi}_j}$  act as shifting operators, namely,

$$\mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}} = b_{\mathbf{m},j}\sigma_{\mathbf{m}_j} \text{ and } \mathbf{Q}_{\bar{\chi}_j}\sigma_{\mathbf{m}} = c_{\mathbf{m},j}\sigma_{\mathbf{m}^j} \quad (13)$$

for some coefficients  $b_{\mathbf{m},j}$  and  $c_{\mathbf{m},j}$ .

We can use Dirac's quantization conditions

$$[\mathbf{Q}_{\chi_j}, \mathbf{Q}_{\chi_k}] = i\hbar \mathbf{Q}_{\{\chi_j, \chi_k\}} \quad \text{and} \quad [\mathbf{Q}_{\chi_j}, \mathbf{Q}_{\bar{\chi}_k}] = i\hbar \mathbf{Q}_{\{\chi_j, \bar{\chi}_k\}} \quad (14)$$

and the identification

$$\mathbf{Q}_{\chi_j}^\dagger = \mathbf{Q}_{\bar{\chi}_j} \quad (15)$$

to determine the coefficients  $b_{\mathbf{m},j}$  and  $c_{\mathbf{m},j}$ , which must satisfy the consistency conditions:

$$b_{\mathbf{m},j} = 0 \text{ if } \mathbb{T}_{\mathbf{m}_j} = \emptyset \quad \text{and} \quad c_{\mathbf{m},j} = 0 \text{ if } \mathbb{T}_{\mathbf{m}^j} = \emptyset. \quad (16)$$

The Bohr-Sommerfeld-Heisenberg quantization described here is an extension of the Bohr-Sommerfeld theory. In the Bohr-Sommerfeld-Heisenberg quantization, the Hilbert space  $\mathfrak{H}$  of quantum states is the same as in the Bohr-Sommerfeld theory. However, in the Bohr-Sommerfeld-Heisenberg theory, we can quantize functions that first degree polynomials in  $\chi_k$  and  $\bar{\chi}_k$  with coefficients given by smooth functions of the actions:

$$F(A_1, \dots, A_n) + \sum_{k=1}^n [F_k(A_1, \dots, A_n) \chi_k + \tilde{F}_k(A_1, \dots, A_n) \bar{\chi}_k].$$

The resulting operators on  $\mathfrak{H}$  first degree polynomials in shifting operators. Higher powers of shifting operators are well defined on  $\mathfrak{H}$ , but they need not be quantizations of the corresponding powers of the functions  $f_k$  or  $\bar{f}_k$  (the usual factor ordering problem).

## 7 Examples

### 7.1 1-dimensional harmonic oscillator

The phase space of the 1-dimensional harmonic oscillator is  $P = \mathbb{R}^2$  with coordinates  $(p, q)$  and the symplectic form  $\omega = dp \wedge dq$ . The Hamiltonian is  $H = \frac{1}{2}(p^2 + q^2)$ . In polar coordinates  $(p, q) = (r \cos \varphi, r \sin \varphi)$ , where  $r = \sqrt{p^2 + q^2}$  and  $\varphi = \tan^{-1} \frac{q}{p}$ , we have  $\omega = dH \wedge d\varphi$ . Here  $H = \frac{1}{2}r^2$  is the action variable, while  $\varphi$  is the corresponding angle. The Heisenberg function  $\chi = p - iq = r e^{-i\varphi}$  leads to quantization equivalent to the Bargmann quantization [1]. It should be noted that  $r = \sqrt{2H}$  is not a smooth function of  $H$ , but  $\chi$  is in  $C^\infty(P)$ . For full details see [6].

## 7.2 Coadjoint orbits of $SO(3)$

Following Souriau [13] we use the presentation of coadjoint orbits of  $SO(3)$  spheres  $S_r^2 = \{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = r^2\}$  endowed with a symplectic form  $\omega = \frac{1}{r} \text{vol}_{S_r^2}$ , where  $\text{vol}_{S_r^2}$  is the standard area form on  $S_r^2$  with  $\int_{S_r^2} \text{vol}_{S_r^2} = 4\pi r^2$ . A coadjoint orbit  $S_r^2$  is quantizable if  $r = \frac{n}{2}\hbar$ , where  $n$  is an integer.

For each  $i = 1, 2, 3$ , we denote by  $J^i$  the restriction of  $x^i$  to the sphere  $S_r^2$ . The functions  $J^1, J^2$  and  $J^3$  are components of the momentum map of the co-adjoint action. They satisfy the Poisson bracket relations  $\{J^i, J^j\} = \sum_{k=1}^3 \varepsilon_{ijk} J^k$ . In spherical polar coordinates

$$J^1 = r \sin \theta \cos \varphi, \quad J^2 = r \sin \theta \sin \varphi, \quad J^3 = r \cos \theta,$$

and

$$\omega = r \sin \theta d\varphi \wedge d\theta = -d(r \cos \theta d\varphi) = d(J^3 d(-\varphi)).$$

Thus,  $(J^3, -\varphi)$  are action-angle coordinates for an integrable system  $(J^3, S_r^2, \omega)$ . In this case, a Heisenberg function is  $\chi = J_+ = \sqrt{r^2 - (J^3)^2} e^{i\varphi}$ , and the resulting Bohr-Sommerfeld-Heisenberg quantization leads to the irreducible unitary representation of  $SO(3)$  corresponding to the co-adjoint orbit  $S_r^2$ . For more details, see [6]. The presented treatment closely resembles the approach of Schwinger [10].

## 7.3 2-dimensional harmonic oscillator

The configuration space of the 2-dimensional harmonic oscillator is  $\mathbb{R}^2$  with coordinates  $x = (x^1, x^2)$ . The phase space is  $T^*\mathbb{R}^2 = \mathbb{R}^4$  with coordinates  $(x, y) = (x^1, x^2, y^1, y^2)$  and canonical symplectic form  $\omega = d(y^1 dx^1 + y^2 dx^2)$ . The Hamiltonian function of the 2-dimensional harmonic oscillator is

$$H(x, y) = \frac{1}{2}((x^1)^2 + (x^2)^2) + \frac{1}{2}((y^1)^2 + (y^2)^2).$$

Orbits of the Hamiltonian vector field  $X_H$  of  $H$  are periodic of period  $2\pi$ . The function  $L(x, y) = x^1 y^2 - x^2 y^1$  generates an action of  $S^1$  on  $T^*\mathbb{R}^2$  that preserves the Hamiltonian  $H$ . Hence,  $(H, L, T^*\mathbb{R}^2, \omega)$  is a completely integrable system. Let

$$\begin{aligned} x_1 &= \frac{-1}{\sqrt{2}}(r_1 \cos \vartheta_1 + r_2 \cos \vartheta_2) & y_1 &= \frac{1}{\sqrt{2}}(r_1 \sin \vartheta_1 + r_2 \sin \vartheta_2) \\ x_2 &= \frac{1}{\sqrt{2}}(-r_1 \sin \vartheta_1 + r_2 \sin \vartheta_2) & y_2 &= \frac{1}{\sqrt{2}}(-r_1 \cos \vartheta_1 + r_2 \cos \vartheta_2). \end{aligned} \tag{17}$$



be a change of coordinates from rectangular  $(x, y)$  variables to polar variables  $(r_1, r_2, \vartheta_1, \vartheta_2)$ . A computation shows that  $H(r, \vartheta) = \frac{1}{2}(r_1^2 + r_2^2)$  and  $L(r, \vartheta) = \frac{1}{2}(r_1^2 - r_2^2)$  and that the change of coordinates (17) pulls back the symplectic form  $\omega = dy_1 \wedge dx_1 + dy_2 \wedge dx_2$  to the symplectic form  $\Omega = d(\frac{1}{2}r_1^2) \wedge d\vartheta_1 + d(\frac{1}{2}r_2^2) \wedge d\vartheta_2$ . Let  $A_1 = \frac{1}{2}r_1^2 = \frac{1}{2}(E(r, \vartheta) + L(r, \vartheta)) \geq 0$  and  $A_2 = \frac{1}{2}r_2^2 = \frac{1}{2}(E(r, \vartheta) - L(r, \vartheta)) \geq 0$ . Then  $(A_1, A_2, \vartheta_1, \vartheta_2)$  with  $A_1 > 0$ , and  $A_2 > 0$  and symplectic form  $\Omega = dA_1 \wedge d\vartheta_1 + dA_2 \wedge d\vartheta_2$  are real analytic action-angle coordinates for the 2-dimensional harmonic oscillator. These coordinates extend real analytically to the closed domain  $A_1 \geq 0$  and  $A_2 \geq 0$ . The Heisenberg functions  $\chi_1 = r_1 e^{i\vartheta_1}$  and  $\chi_2 = r_2 e^{i\vartheta_2}$  give rise to the Bohr-Sommerfeld-Heisenberg quantization of the 2-dimensional harmonic oscillator. For more details see [5].

## 7.4 Mathematical Pendulum

The phase space of the mathematical pendulum is  $T^*S^1$  with coordinates  $(p, \alpha)$  and symplectic form  $\omega = dp \wedge d\alpha$ . The Hamiltonian of the system is  $H = \frac{1}{2}p^2 - \cos \alpha + 1$ . The Hamiltonian system  $(H, T^*S^1, \omega)$  violates Assumption 2, because for  $H > 2$ , level sets of the Hamiltonian  $H$  have two connected components. We are investigating how to extend to this case the theory presented here.

## References

- [1] V. Bargmann, Remarks on a Hilbert space of analytic functions, *Proc. Nat. Acad. Sci. U.S.A* **48** (1962) 199–204.
- [2] N. Bohr, “On the constitution of atoms and molecules” (Part I), *Philosophical Magazine*, **26** (1913) 1-25.
- [3] M. Born and P. Jordan, “Zur Quantenmechanik”, *Zeit. f. Phys.*, **34** (1925) 858–888.
- [4] R. Cushman and J. Duistermaat, The quantum mechanical spherical pendulum, *Bull. Amer. Math. Soc.* **19** (1988) 475–479.
- [5] R. Cushman and J. Śniatycki, Bohr-Sommerfeld-Heisenberg quantization of the 2-dimensional harmonic oscillator, *Arxiv:mathSG*. 1207-1477.

- [6] R. Cushman and J. Śniatycki, Bohr-Sommerfeld-Heisenberg theory in geometric quantization, *J. Fixed Point Theory and Appl.* **13** (2013) 3–24.
- [7] P.A.M. Dirac, The fundamental equations of quantum mechanics, *Proc. Roy. Soc. London*, **A 109** (1925) 642–653.
- [8] P.A.M. Dirac, On the theory of quantum mechanics, *Proc. Roy. Soc. London*, **A 112** (1926) 661–677.
- [9] W. Heisenberg, Über die quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen, *Zeit. f. Phys.*, 33 (1925) 879–893.
- [10] J. Schwinger, In: *Quantum theory of angular momentum: a collection of reprints and original papers*, eds. L.C. Biedernharn and H. van Dam, On angular momentum, p.229, Academic Press, New York, 1965.
- [11] J. Śniatycki, *Geometric quantization and quantum mechanics*, Applied Mathematical Sciences **30** Springer-Verlag, New York-Berlin, 1980.
- [12] A. Sommerfeld, Zur Theorie der Balmerischen Serie, *Sitzungsberichte der Bayerischen Akademie der Wissenschaften (München), mathematisch-physikalische Klasse*, (1915) 425–458.
- [13] J.-M. Souriau, 1975 Lecture at M.I.T.